FLOW PAST A PROFILE IN A BOUNDED SUPERSONIC STREAM

(OBTEKANIE PROFILIIA V SVERKHZVUKOVOM STESNENNOM POTOKE)

PMM Vol.22, No.6, 1958, pp.815-818

M.D. KHASKIND and V.S. KHOMENKO
(Odessa)

(Received 28 March 1958)

We consider the nonlinear two-dimensional problem of steady flow past a wing profile with a sharp leading edge in a bounded supersonic stream of isentropic gas. The boundaries of the stream are taken to be rectilinear and parallel, and the profile is placed unsymmetrically with respect to them.

The solution is based on the use of the Legendre transformation, by means of which the equations of gas motion are transformed into symmetric linear equations. Furthermore, a simple approximation is made to the Bernoulli integral, and the complete problem is reduced to the solution of a finite system of functional equations, the properties of which are studied in detail. On the basis of these properties, concrete calculations are carried out for the aerodynamic force and moment acting on the airfoil. The results obtained indicate a considerable sensitivity of the solution to nonlinear effects in a bounded stream even at low angles of attack.

The general equations of steady plane and irrotational motion of an isentropic gas have the form

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \qquad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \tag{1}$$

$$M^2 = \frac{1}{\nu^{\gamma-1}}\left(M_\infty{}^2 + \frac{2}{\gamma-1}\right) - \frac{2}{\gamma-1} \tag{2}$$

$$M = \frac{w}{a}, \quad a^2 = \gamma\frac{p_\infty}{\rho_\infty}\nu^{\gamma-1}, \quad \nu = \frac{\rho}{\rho_\infty}, \quad a_\infty{}^2 = \gamma\frac{p_\infty}{\rho_\infty}, \quad M_\infty = \frac{u_\infty}{a_\infty} \tag{3}$$

where w is the velocity of the flow with components u and v along the axes, ρ - the density of the gas, γ - the adiabatic exponent, a - the velocity of sound, and the index ∞ denotes the properties of the undisturbed flow, in which the pressure is p_∞.

To linearize equations (1) we use the Legendre transformation

$$\varphi + \frac{i}{\rho}\,\psi = zwe^{-i\theta} - \left(\Phi + \frac{i}{\rho}\,\Psi\right) \qquad \left(z = x + iy, \quad i = \sqrt{-1}\right) \tag{4}$$

Here ϕ and ψ are respectively the velocity potential and stream function, θ the angle of inclination of the velocity vector to the x-axis, and Φ Ψ the functions of the Legendre transformation.

Substitution of (4) into (1) with consideration of relations (3) leads to the following linear symmetrical equations

$$\frac{\partial\Phi}{\partial\sigma} = \frac{K}{\rho_\infty}\frac{\partial\Psi}{\partial\theta}, \qquad \frac{\partial\Phi}{\partial\theta} = \frac{K}{\rho_\infty}\frac{\partial\Psi}{\partial\sigma} \tag{5}$$

where σ and K are determined by the expressions

$$\sigma = \frac{\pi}{2} - \int_0^\nu \frac{K d\nu}{1+\nu^2K^2}, \; M^2 = \frac{C}{\nu^{\gamma-1}}\exp\left(-\int_0^\nu \frac{K^2 d\nu^2}{1+\nu^2K^2}\right), C = M_\infty^{\,2}\exp\left(\int_0^1 \frac{K^2 d\nu^2}{1+\nu^2K^2}\right) \tag{6}$$

where on the basis of (2) σ is a function of M.

Henceforth we assume

$$\frac{1}{K^2} = M_\infty^{\,2} - 1 = \tan^{-2}\sigma_\infty \tag{7}$$

and according to (6) we obtain

$$M^2 = \frac{M_\infty^{\,4}}{\nu^{\gamma-1}(M_\infty^{\,2} - 1 + \nu^2)}, \qquad \nu = \tan\sigma_\infty \cot\sigma \tag{8}$$

With such a choice of K the first formula (8) gives an approximate expression for equation (2) such that the values of (2) and (8) together with their derivatives coincide at the point $M = M_\infty$. In Fig. 1 the exact and approximate dependence of ν on M, determined from (2) and (8) with $\gamma = 1.405$, are shown by the solid and dashed lines respectively. We note that the existing ways of approximating the equation of state of the gas with the use of different kinds of general transformations [1-6] are very accurate, but the transformation to the physical plane is then made very complicated and the solution of the given problem encounters great difficulty. However, with the case considered here the transformation from the $\sigma\theta$ to the xy-plane is determined on the basis of (4), (5) and (8) by means of the formulas

$$\begin{aligned}(x\cos\theta + y\sin\theta)\cos\sigma &= \Phi_1'(\sigma+\theta) + \Phi_2'(\sigma-\theta)\\ (x\sin\theta - y\cos\theta)\sin\sigma &= -\Phi_1'(\sigma+\theta) + \Phi_2'(\sigma-\theta)\end{aligned} \tag{9}$$

where $\Phi_1(\sigma+\theta)$ and $\Phi_2(\sigma-\theta)$ are arbitrary functions satisfying equations (5) with $K = \text{const}$.

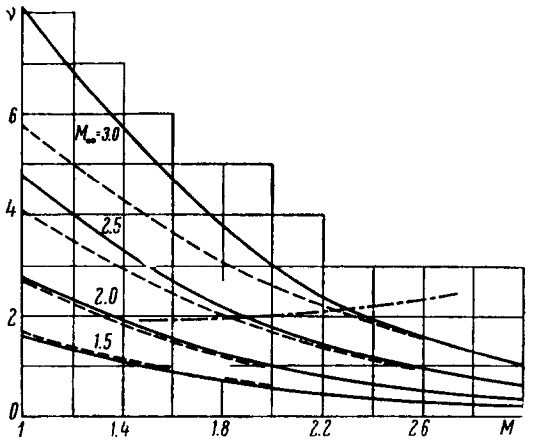

Fig. 1.

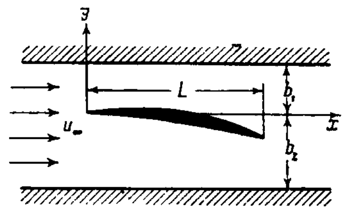

Fig. 2.

We now consider the boundary conditions. For $x < 0$ (Fig. 2) the supersonic stream is undisturbed and consequently

$$\begin{gathered} w = u_\infty \quad \text{for } y = 0,\ x < 0, \\ \theta = \theta(x/L) \quad \text{on} \quad C_0 \\ \theta = 0 \text{ for } y = b_1,\ y = -b_2 \end{gathered} \tag{10}$$

where $\theta(x/L)$ is a given function on the profile C_0.

The solution of the problem in the regions above and below the profile is carried out separately and in the same way. We therefore limit ourselves to consideration of the upper region only.

From the condition $\theta = 0$ at $y = b_1$ we at once find that $\Phi_1'(\sigma) = \Phi_2'(\sigma) + b_1 \sin\sigma$. Satisfying conditions (10) completely gives

$$\begin{gathered} t_0 \cos[\sigma(t_0) - \theta(t_0)] - \tau_0 \sin[\sigma(t_0) - \theta(t_0)] = \frac{2}{L}\Phi_2'[\sigma(t_0) - \theta(t_0)] \\ t_0 \cos[\sigma(t_0) + \theta(t_0)] - (2\zeta_1 - \tau_0)\sin[\sigma(t_0) + \theta(t_0)] = \frac{2}{L}\Phi_2'[\sigma(t_0) + \theta(t_0)] \\ \left(t_0 = \frac{x}{L},\ \tau_0 = \frac{y(t_0)}{L},\ \zeta_1 = \frac{b_1}{L} \right) \end{gathered} \tag{11}$$

where $y(t_0)$ is the equation of the upper part of the profile.

In particular, with $\zeta_1 = \infty$ in the second equation (11) we find the relation

$$\sigma(t_0) + \theta(t_0) = \sigma_\infty \tag{12}$$

which determines the Prandtl-Meyer solution for an unbounded supersonic stream.

To solve equation (11) in the general case we introduce the auxiliary variable t_1, determined by the condition

$$\sigma(t_0) + \theta(t_0) = \sigma(t_1) - \theta(t_1) \tag{13}$$

Then the function Φ_2' is eliminated from equation (11) and we obtain

$$t_1 = t_0 - (2\zeta_1 - \tau_0 - \tau_1)\,\mathbf{tan}\,[\sigma(t_0) + \theta(t_0)], \qquad \tau_1 = \tau_0(t_0) \tag{14}$$

Expressions (13) and (14) are equations for the determination of σ. The general solution of these equations can be obtained by means of a method of successive approximations, using (12) as the zero approximation:

$$\sigma_0(t_1) + \theta(t_1) = \sigma_\infty, \qquad \sigma_p(t_0) + \theta(t_0) = \sigma_{p-1}(t_1) - \theta(t_1) \qquad (p = 1, 2, \ldots,) \tag{15}$$

As a result we obtain

$$\sigma(t_0) = \sigma_\infty - \theta(t_0) - 2\sum_{k=1}^{\infty} \theta(t_k) \tag{16}$$

where t_p is related to t_{p-1} by formula (14), that is

$$t_p = t_{p-1} - (2\zeta_1 - \tau_{p-1} - \tau_p)\,\mathbf{tan}\,[\sigma(t_{p-1}) + \theta(t_{p-1})] \quad (\tau_p = \tau_0(t_p),\ p = 1, 2, \ldots) \tag{17}$$

Let the quantity σ be eliminated by joint considerations of (13) and (14), i.e. $t_1(t_0) = Gt_0$, where G is a certain functional operator; then from (17) we have that $t_p(t_0) = G^p t_0$. Consequently the sequence of values t_p satisfies the condition $t_p(t_0) = G^{p-k} t_k(t_0)$, since $t_k(t_0) = G^k t_0$. Using this and substituting (16) into (17), we find

$$t_p = t_{p-1} - (2\zeta_1 - \tau_{p-1} - \tau_p)\ \mathbf{tan}\left[\sigma_\infty - 2\sum_{k=p}^{\infty} \theta(t_k)\right] \qquad (p = 1, 2, \ldots) \tag{18}$$

We consider the values $0 < (t_k) + \theta(t_k) < \pi/2$ $(k = 0, 1, \ldots)$ for $0 \leqslant t_0 \leqslant 1$; then from (17) we obtain (m = min tan $(\sigma + \theta)$)

$$t_p \leqslant t_{p-1} - (2\zeta_1 - \tau_{p-1} - \tau_p)\,m, \qquad t_p < t_{p-1} \qquad (p = 1, 2, \ldots) \tag{19}$$

The sequence of values t_p thus decreases with increasing p. Therefore, for $p > n$ we obtain $t_p < 0$ $(\theta(t_p) = 0)$ and the series in (16) and (18) are transformed into the finite sums*

---

* The connection between equations (20) and (21) (determination of direct and reflected waves, points of reflection of characteristics, and so on) was studied in detail by Khomenko and applied to the investigation of the motion of a ship in a shallow-water canal.

$$t_p = t_{p-1} - (2\zeta_1 - \tau_{p-1} - \tau_p) \tan \left[\sigma_\infty - 2 \sum_{k=p}^{n} \theta(t_k)\right] \qquad (p = 1,2,\ldots,n) \tag{20}$$

$$\sigma(t_0) = \sigma_\infty - \theta(t_0) - 2 \sum_{k=1}^{n} \theta(t_k) \tag{21}$$

To solve equation (20) we pick a positive integer *n*, for which we find the point of reflection of the characteristic on the profile and the interval of the velocities (σ_∞) corresponding to the given *n*. Then the values t_p are determined from (20) in the form of a sequence, namely t_n, t_{n-1}, etc. to $t_0 = t_0(t_n)$. We determine the pressure distribution on the profile from the equation of state

$$p = p_\infty \nu^\gamma = p_\infty \tan^\gamma \sigma_\infty \cot^\gamma \sigma(t_0) \tag{22}$$

and we can thus find the integrated values of aerodynamic force and of moment with respect to the leading edge of the profile. In particular, for a flat plate placed at an angle of attack θ = const the dimensionless coefficients of lift and moment are determined in the form

$$c_p = \frac{P}{{}^1/_2\,\rho_\infty L_1 u_\infty^2} = \frac{2}{M_\infty^{2\gamma}} \int_0^1 (\nu_-^\gamma - \nu_+^\gamma)\, dt_0$$

$$c_m = \frac{m}{{}^1/_2\,\rho_\infty L_1^2 u_\infty^2} = \frac{2}{M_\infty^{2\gamma}} \int_0^1 (\nu_-^\gamma - \nu_+^\gamma)\, t_0 dt_0 \qquad \left(L_1 = \frac{l}{\cos\theta}\right) \tag{23}$$

where L_1 is the width of the plate and *P* and *m* the lift and moment.

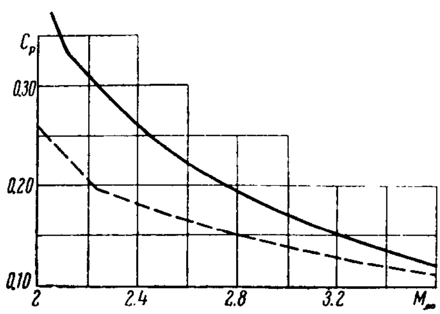

Fig. 3.

In the case just considered, of a flat plate, $r_0 = -\ t_0 \tan\theta$ and use of (20)-(23) leads to integrated expressions such that $\sigma(t_0)$ is constant between two neighboring points of reflection of characteristics and undergoes a jump in passing through these points. We compare the results obtained from nonlinear theory with those of linearized theory. In the linear case θ and r_0 assume small values and equation (20) simplifies. We have

$$t_p = t_0 - 2p\zeta_1 \tan\sigma_\infty \qquad (p = 1, \ldots, n) \tag{24}$$

Simplifying the expression (22) to the first order of small quantities, taking account of (21), (3), and (7), we obtain

$$p = p_\infty + \frac{\rho_\infty u_\infty^2}{\sqrt{M_\infty^2 - 1}} \left[\theta(t_0) + 2 \sum_{k=1}^{n}{}' \theta(t_k)\right] \tag{25}$$

If we perform the linearization on equations (1) and (2), we obtain just the same expressions (24) and (25) (see, for example [7]).

In Figs. 3-5 is shown the dependence of c_p, c_m and $l_0 = c_m/c_p$ on M_∞ for $\theta = 0.1$ rad, $\zeta_1 = b_2/L = 0.3$ calculated from formula (23) (solid line) and according to linearized theory (dashed line).

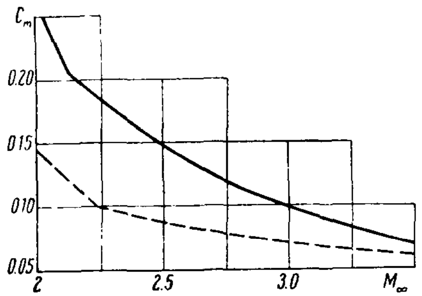

Fig. 4.

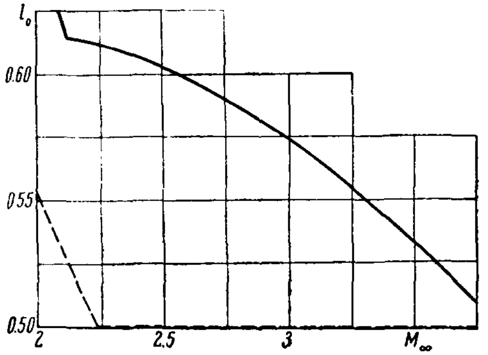

Fig. 5.

In the example considered, the lower limit for M is represented by the dot-dash line in Fig. 1. Beginning with this value, we have good agreement between formulas (2) and (8). From a comparison of the values in Figs. 3-5 it is evident that the solution of the problem possesses a strong sensitivity to nonlinear effects in a bounded stream and that the values of c_p and c_m of nonlinear theory significantly exceed the values from linearized theory even at small angles of attack.

This method permits calculation of the aerodynamic characteristics of a profile in a supersonic stream near the surface of the earth.

BIBLIOGRAPHY

1. Khristianovich, S.A., Priblizhennoe integrirovanie uravnenii sverkh-zvukovogo techeniia gaza (Approximate integration of the equations of supersonic gas flow). *PMM* Vol. 11, No. 2, 1947.

2. Sedov, L.I., *Ploskie zadachi gidrodinamiki i aerodinamiki (Plane Problems in Hydrodynamics and Aerodynamics)*. Gostekhizdat, 1950.

3. Vallander, S.V., Ob integrirovanii giperbolicheskoi sistemy dvukh uravnenii pri dvukh nezavisimykh peremennykh (On the integration of a hyperbolic system of two equations in two independent variables). *Dokl. Akad. Nauk SSSR* Vol. 83, No.5, 1952.

4. Dombrovskii, G.A., O priblizhennom integrirovanii sverkhzvukovogo techeniia gaza (On the approximate integration of supersonic gas flow). Collected papers on "Theoretical hydrodynamics" under the editorship of L.I. Sedov, No. 11, issue 3. Oborongiz, 1953.

5. Dombrovskii, G.A., Priblizhennoe analiticheskoe reshenie ploskoi zadachi sverkhzvukovoi gazavoi dinamiki (Approximate analytic solution of plane problems of supersonic gas dynamics). Collected papers on "Theoretical hydrodynamics" under the editorship of Academician L.I. Sedov, No. 12, issue 4. Oborongiz, 1954.

6. Rudnev, Iu.V., O priblizhennom integrirovanii uravnenii gazovoi dinamiki (On approximate integration of the equations of gas dynamics). Collected papers on "Theoretical hydrodynamics" under the editorship of Academician L.I. Sedov, No. 16, issue 6, Oborongiz, 1955.

7. Khomenko, V.X., Dvizhenie sudna v melkovodnom kanale (Motion of a ship in a shallow-water canal). *Izv. Akad. Nauk SSSR, OTN*, No. 10, 1957.